WNE Linear Algebra
Final Exam
Series A

6 February 2021

Questions

Please use a single file for all questions. Give reasons to your answers or provide a counterexample. Please provide the following data in the pdf file

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Each question is worth 4 marks.

Question 1.
Let $q\colon \mathbb{R}^2 \to \mathbb{R}$ be a quadratic form given by a symmetric matrix $M \in M(2 \times 2; \mathbb{R})$, i.e., $M = M^\top$ and

$$q((x, y)) = \begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix}.$$

If $\det M \neq 0$ does it follow that for all vectors $(x, y) \in \mathbb{R}^2$, $(x, y) \neq \mathbf{0}$

$$q((x, y)) \neq 0?$$

Answer 1.
No, it does not. Consider

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$q((1, 0)) = \begin{bmatrix} 1 & 0 \end{bmatrix} M \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

Question 2.
Let $V \subset \mathbb{R}^3$ be a proper subspace of $\mathbb{R}^3$ (i.e., $\dim V = 1$ or $\dim V = 2$). Let

$$P_{V^\perp}\colon \mathbb{R}^3 \to \mathbb{R}^3,$$

denote the (linear) orthogonal projection onto $V^\perp$ (i.e. onto the orthogonal complement of the subspace V) and let

$$S_V\colon \mathbb{R}^3 \to \mathbb{R}^3,$$

denote the (linear) orthogonal reflection about V. Does it follow that

$$S_V \circ P_{V^\perp} = -P_{V^\perp}?$$

Answer 2.
Yes, it does. For any $v \in \mathbb{R}^3$ let

$$v = w + u,$$

where $w \in V$ and $u \in V^{\perp}$. Then

$$P_{V^{\perp}}(v) = u,$$
$$S_V(u) = -u,$$

hence

$$S_V(P_{V^{\perp}}(v)) = -u = -P_{V^{\perp}}(v).$$

Question 3.
If $A, B \in M(2 \times 2; \mathbb{R})$ does it follow that

$$\det(A^2 + 2AB + B^2) \geqslant 0?$$

Answer 3.
No, it does not. Consider

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$A^2 + 2AB + B^2 = 2AB = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix},$$

that is $\det(A^2 + 2AB + B^2) = -4 < 0$ (note that $AB \neq BA$).

Question 4.
Let $A, B \in M(m \times n; \mathbb{R})$ be matrices such that

$$A \xrightarrow{r_i + \alpha r_j} B,$$

for some $i, j \in \{1, 2, \ldots, m\}$ and some $\alpha \in \mathbb{R}$ (i.e., one is obtained from the other by the row operation $r_i + \alpha r_j$). Does it follow that the reduced (row) echelon form of matrix A is the same as the reduced (row) echelon form of matrix B?

Answer 4.
Yes, it does. Row equivalent matrices have the same reduced row echelon form (cf. Lecture 1).

Question 5.
Let $A, B \in M(2 \times 2; \mathbb{R})$ be two matrices and let $v \in \mathbb{R}^2$ be an eigenvector of matrix A and simultaneously an eigenvector of matrix B, corresponding to the same eigenvalue. Is vector $v \in \mathbb{R}^2$ an eigenvector of matrix $AB - B^2$?

Answer 5.
Yes, it is an eigenvector corresponding to the eigenvalue 0,

$$(AB - B^2)v = A(Bv) - B(Bv) = A(\lambda v) - B(\lambda v) = \lambda^2 v - \lambda^2 v = 0v.$$